A complete linear basis for (machine) learning jet substructure

Machine Learning for Jet Physics Workshop, 2017

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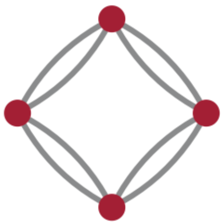
Based on work with Patrick T. Komiske and Jesse Thaler

December 12, 2017



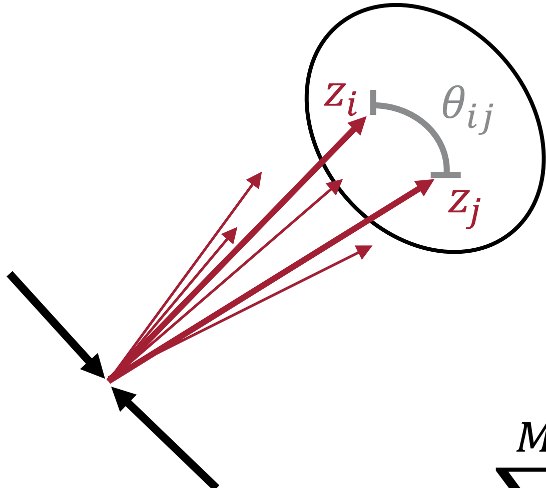
Energy Flow Polynomials (EFPs)

The Energy Flow Basis from IRC safety

Taming the (IRC-safe) Substructure Zoo

Spanning Substructure with Linear Regression

Anatomy of an Energy Flow Polynomial:



Energy Fraction | Pairwise Angular Distance

$$e^+e^-: \; z_i = \frac{E_j}{\sum_k E_k}, \quad \theta_{ij} = \left(\frac{2p_i^\mu p_{j\mu}}{E_i E_j}\right)^{\frac{\beta}{2}}$$

$$\text{Hadronic:} \; z_i = \frac{p_{Tj}}{\sum_k p_{Tk}}, \quad \theta_{ij} = \left(\Delta y_{ij}^2 + \Delta\phi_{ij}^2\right)^{\frac{\beta}{2}}$$

In equations:

$$\mathrm{EFP}_G = \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \cdots \sum_{i_N=1}^{M} z_{i_1} z_{i_2} \cdots z_{i_N} \prod_{(k,l)\in G} \theta_{i_k i_l}$$

multigraph

In words:

Correlator — Sum over all *N*-tuples of particle in the event

of **Energies** — Product of the *N* energy fractions

and **Angles** — One $\theta_{i_k i_l}$ for each edge in $(k,l) \in G$

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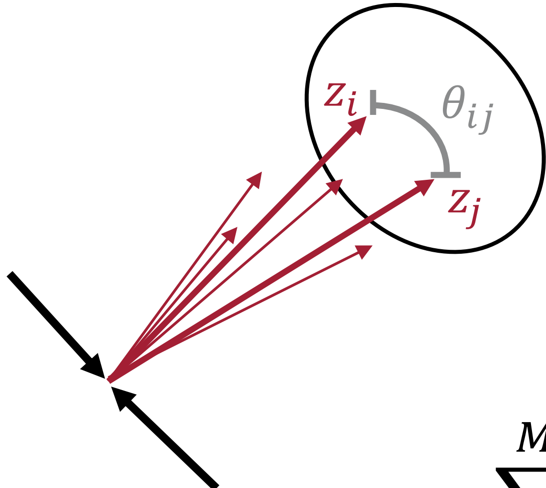

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In words:

Correlator of Energies and Angles

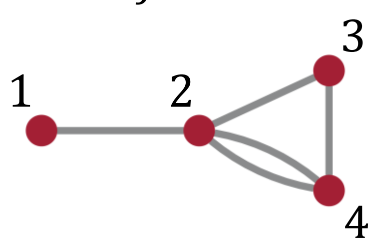
- **Correlator**: Sum over all N-tuples of particle in the event
- **Energies**: Product of the N energy fractions
- **Angles**: One $\theta_{i_k i_l}$ for each edge in $(k,l) \in G$

In pictures:

$\longleftrightarrow \; z_{i_j}$

$\longleftrightarrow \; \theta_{i_k i_l}$

(e.g.)

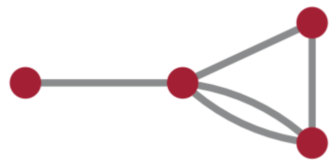


$$= \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \sum_{i_3=1}^{M} \sum_{i_4=1}^{M} z_{i_1} z_{i_2} z_{i_3} z_{i_4} \, \theta_{i_1 i_2} \theta_{i_2 i_3} \theta_{i_3 i_4} \theta_{i_2 i_4}^2$$

(any index labelling works)

Multigraph/EFP Correspondence

Multigraph ⟷ EFP



$$= \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \sum_{i_3=1}^{M} \sum_{i_4=1}^{M} z_{i_1} z_{i_2} z_{i_3} z_{i_4} \, \theta_{i_1 i_2} \theta_{i_2 i_3} \theta_{i_3 i_4} \theta_{i_2 i_4}^2$$

j ⟷ z_{i_j}

k — l ⟷ $\theta_{i_k i_l}$

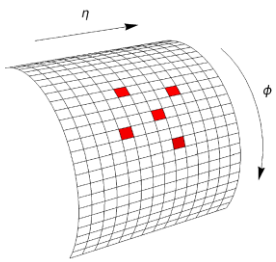
N	Number of vertices	⟷	*N*-particle correlator
d	Number of edges	⟷	Degree of angular monomial
χ	Treewidth + 1	⟷	Optimal VE Complexity

e.g. Tree graph EFPs are $O(M^2)$! Surprisingly efficient to compute. Stay tuned… See [P. Komiske's talk].

Connected	⟷	Prime
Disconnected	⟷	Composite

⋮

Energy Flow Polynomials (EFPs)



The Energy Flow Basis from IRC safety

Taming the (IRC-safe) Substructure Zoo

Spanning Substructure with Linear Regression

EFPs linearly span IRC-safe observables

IRC-safe Observable

EFPs linearly span IRC-safe observables

IRC-safe Observable

Energy Expansion: Expand/approximate the observable in polynomials of the particle energies

IR safety: Observable unchanged by addition of infinitesimally soft particle

C safety: Observable unchanged by the collinear splitting of a particle

Relabeling Symmetry: All ways of indexing particles are equivalent

New, direct argument from IRC safety
See also: F. Tkachov, hep-ph/9601308
N. Sveshnikov and F. Tkachov, hep-ph/9512370

Energy correlators linearly span IRC-safe observables

EFPs linearly span IRC-safe observables

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Energy correlators linearly span IRC-safe observables

Angular Expansion: Expansion/approximation of angular part of correlators in pairwise angular distances
Analyze: Identify the unique analytic structures that emerge as non-isomorphic multigraphs/EFPs

Similar expansions & emergent multigraphs in:
M. Hogervorst *et al.* arXiv:1409.1581
B. Henning *et al.* arXiv:1706.08520

EFPs linearly span/approximate IRC-safe observables!

Organization of the basis

EFPs are truncated by angular degree *d*, the order of the angular expansion.

Finite number at each order in *d*!
All prime EFPs up to *d=5* →

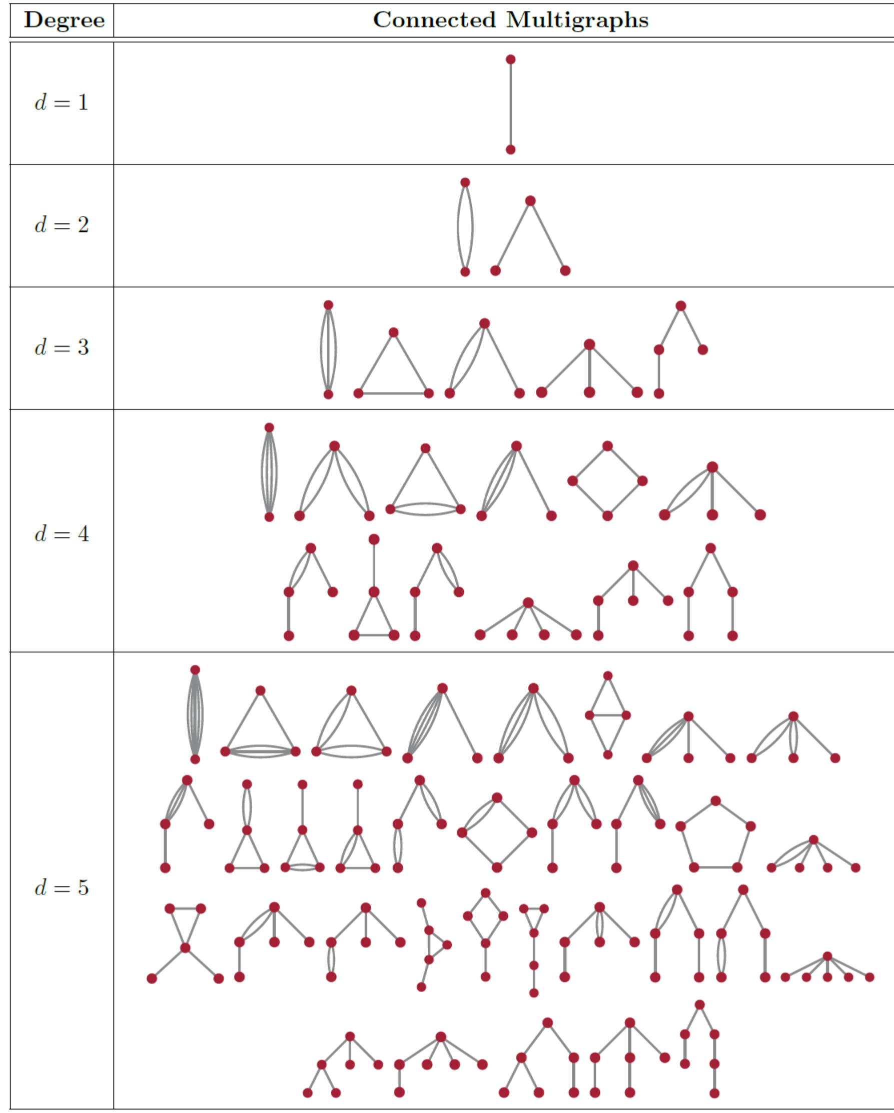

Image files for all of the prime EFP multigraphs up to d = 7 are available here.

Online Encyclopedia of Integer Sequences (OEIS)

A050535 # of multigraphs with *d* edges
# of EFPs of degree *d*

A076864 # of connected multigraphs with *d* edges
# of prime EFPs of degree *d*

↓

Exactly 1000 EFPs up to degree *d=7!*

Energy Flow Polynomials (EFPs)

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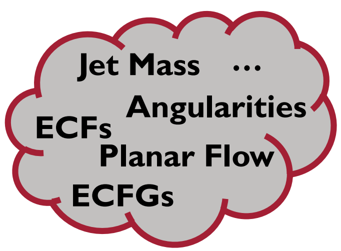

Taming the (IRC-safe) Substructure Zoo

Spanning Substructure with Linear Regression

Jet Observables with Energy Flow

Jet Mass

Dumbbell EFP

$$\frac{m_J^2}{p_{TJ}^2} = \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} z_{i_1} z_{i_2} (\cosh \Delta y_{i_1 i_2} - \cos \Delta \phi_{i_1 i_2}) = \frac{1}{2}$$

$$+ \cdots$$

Can include these using a fully general measure

Jet Observables with Energy Flow

Jet Mass
Dumbbell EFP

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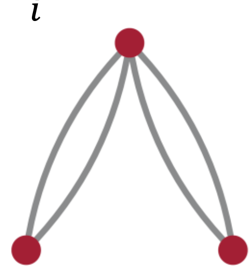
Can include these using a fully general measure

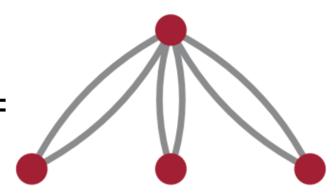
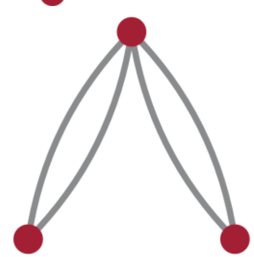
Angularities
Star Graph EFPs

$$\lambda^{(\alpha)} = \sum_{i}^{M} z_i \theta_i^{\alpha}$$

[C. Berger, T. Kucs, and G. Sterman, hep-ph/0303051]
[L. Almeida, *et al.*, arXiv:0807.0234]
[S. Ellis, *et al.*, arXiv:10010014]
[A. Larkoski, J. Thaler, and W. Waalewijn, arXiv:1408.3122]

$$\lambda^{(4)} =$$  $$-\frac{3}{4}$$ 

(and so on, for all even angularities)

$$\lambda^{(6)} =$$  $$-\frac{3}{2}$$   $$+\frac{5}{8}$$ 

using pT-centroid axis

Jet Observables with Energy Flow

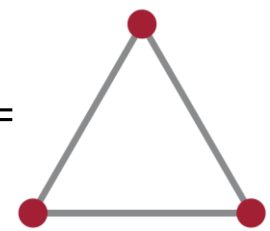
Energy Correlation Functions

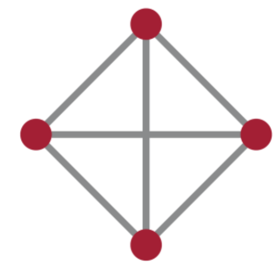
Complete Graph EFPs

$$e_N^{(\beta)} = \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \cdots \sum_{i_N=1}^{M} z_{i_1} z_{i_2} \cdots z_{i_N} \prod_{k<l\in\{1,\cdots,N\}} \theta_{i_k i_l}^{\beta}$$

[A. Larkoski, G. Salam, and J. Thaler, arXiv:1305.0007]

$e_2^{(\beta)} =$ 

$e_3^{(\beta)} =$ 

$e_4^{(\beta)} =$  $\cdots$

with measure choice of β

Jet Observables with Energy Flow

Energy Correlation Functions
Complete Graph EFPs

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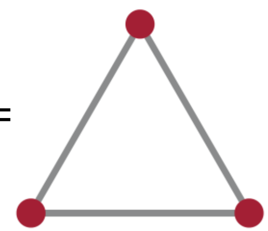
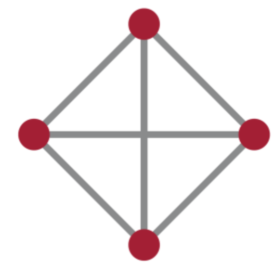
$e_2^{(\beta)} =$  $e_3^{(\beta)} =$  $e_4^{(\beta)} =$  $\cdots$

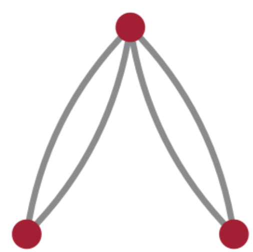
with measure choice of β

Geometric Moments
Higher dumbbell EFPs

$$\mathbf{C} = \sum_{i_1=1}^{M} z_i \begin{pmatrix} \Delta y_i^2 & \Delta y_i \Delta\phi_i \\ \Delta\phi_i \Delta y_i & \Delta\phi_i^2 \end{pmatrix}$$

$$(\text{e.g.})\ \text{Pf} = \frac{4 \det \mathbf{C}}{(\text{tr}\,\mathbf{C})^2}$$

[L. Almeida, *et al.*, arXiv:0807.0234]
[J. Thaler and L.-T. Wang, arXiv:0806.0023]
[J. Gallicchio and M. Schwartz, arXiv:1211.7038]

$\text{tr}\,\mathbf{C} = \frac{1}{2}$  $\det \mathbf{C} =$  $-\frac{1}{2}$ 

using pT-centroid axis

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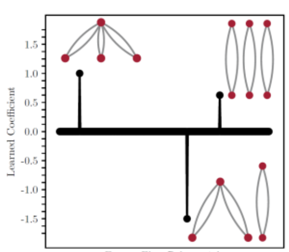


Spanning Substructure with Linear Regression

Linear Models and Energy Flow

$$S = \sum_G w_G \, \mathrm{EFP_G}$$

Machine learn these

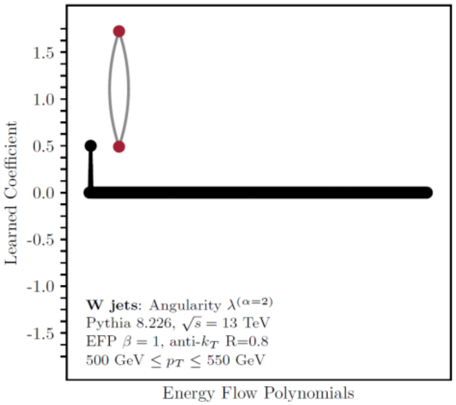
Linear methods:

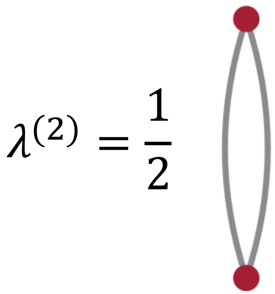
Utilize the linear completeness of the Energy Flow basis.
Convex and few/no hyperparameters to tune.
Achieve global optimum via closed-form solution or convergent iteration.
Simple models with the minimum number of parameters/input.

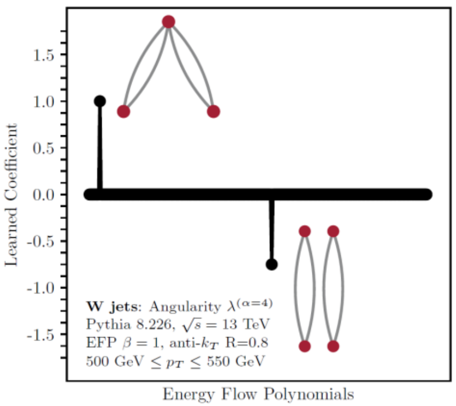
Rich in tools and applications:

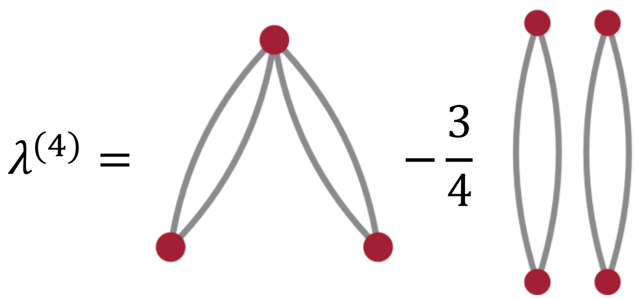
First few chapters of C. Bishop's *Pattern Recognition and Machine Learning*:

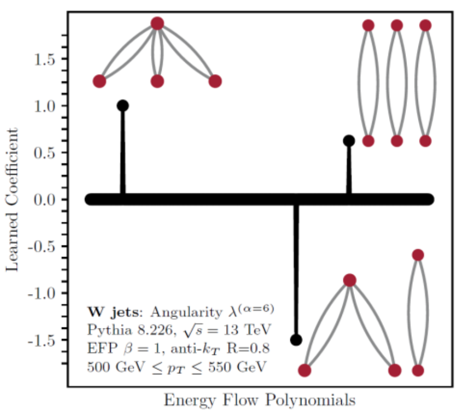
Confirming Analytic Relationships with Regression

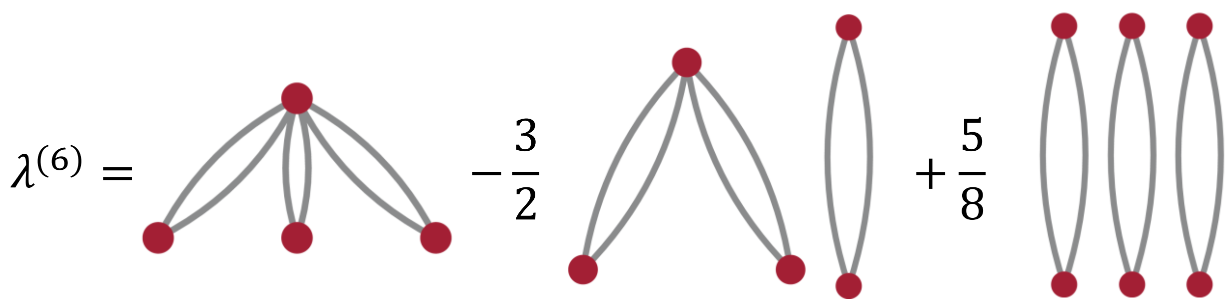












Linear Regression and IRC-safety

$\frac{m_J}{p_{TJ}}$: IRC safe. No Taylor expansion due to square root.

$\lambda^{(\alpha=1/2)}$: IRC safe. No simple analytic relationship.

τ_2: IRC safe. Algorithmically defined.

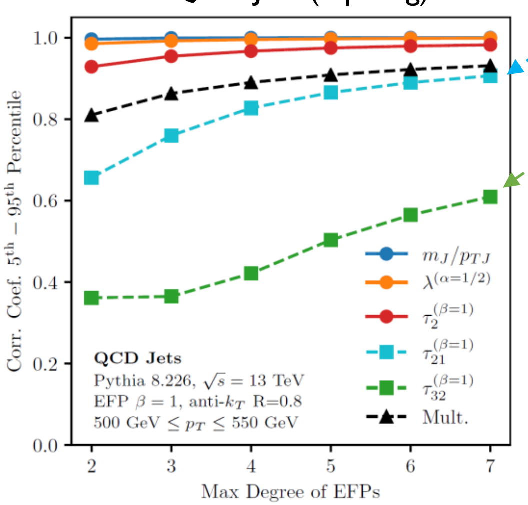
τ_{21}: Sudakov safe. Safe for 2-prong jets and higher.

[A. Larkoski, S. Marzani, and J. Thaler, arXiv:1502.01719]

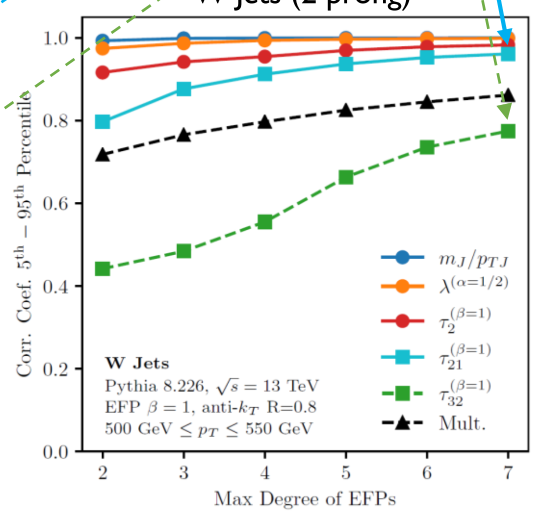
τ_{32}: Sudakov safe. Safe for 3-prong jets and higher.

Multiplicity: IRC unsafe.

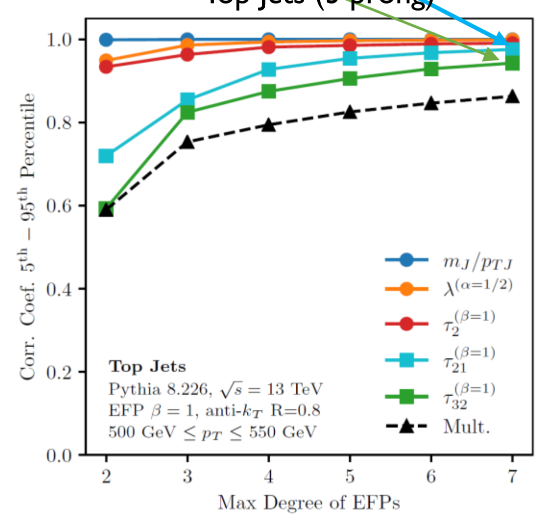
QCD Jets (1 prong)



W Jets (2 prong)



Top Jets (3 prong)



Expected to be IRC safe = Solid.
Expected to be IRC unsafe = Dashed.

Conclusions

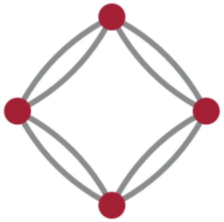
EFPs form a complete, linear representation of the jet

- EFPs energy correlators with monomial angular structure
- Encompass many existing classes of expert variables
- Opens the door to using linear methods for jet substructure
- IRC-unsafe information? Combine!
  - Use EFPs & linearity to reduce radiation pattern to a single optimal observable

(Linear) Learning is easy

- Linear models are convex & even closed-form at times
- Few or no hyperparameters to tune at all
- Guaranteed global optima

The End



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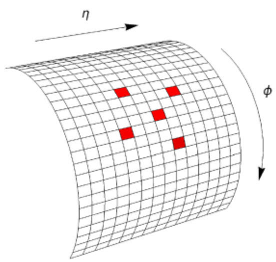

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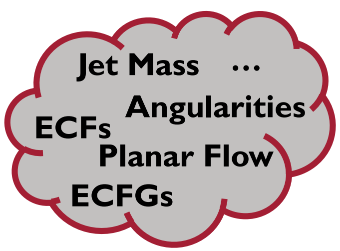

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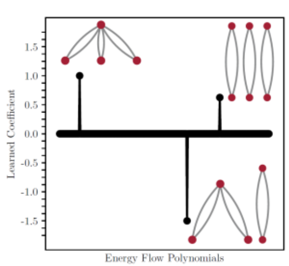

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